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ON THE ENDOMORPHISM RING OF A MODULE NOETHERIAN WITH RESPECT TO A TORSION THEORY

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ABSTRACT

If M is a module torsionfree and noetherian with respect to a torsion theory, if S is the endomorphism ring of M, and if $L(S)$ is the ideal of S consisting of all endomorphisms with large kernels, then $L(S)$ is nilpotent and a bound on the index of nilpotency of $L(S)$ is given.

Let R be an associative ring with unit element 1 and let R-mod be the category of all unitary left R-modules. Morphisms in R-mod will be written as acting on the right. Denote by R-tors the complete brouwerian lattice of all (hereditary) torsion theories on R-mod. Notation and terminology concerning such theories will follow [7]. In particular, if $\tau \in R$-tors then a submodule N of a left R-module M is said to be *τ-dense* [resp. *τ-pure*] in M if and only if M/N is τ-torsion [resp. τ-torsionfree]. We will denote by $T_\tau(-)$ the τ-torsion subfunctor of the identity functor on R-mod. If N is a submodule of a left R-module M, then a submodule N' of M is said to be the *τ-purification* of N in M if and only if $N'/N = T_\tau(M/N)$. A nonzero τ-torsionfree left R-module M is said to be *τ-cocritical* if and only if every nonzero submodule of M is τ-dense in it. The importance of such modules is emphasized in [7]. A left R-module is said to be *τ-noetherian* if and only if it satisfies the ascending chain condition on τ-pure submodules. We see that τ-cocritical left R-modules are trivially τ-noetherian. These modules have been principally studied by Năstăsescu, in a series of papers beginning with [15], and recently by several other authors as well.

One of Năstăsescu's important results states that if M is a τ-torsionfree τ-noetherian quasi-injective left R-module then the endomorphism ring of M is semiprimary [16]. In particular, the Jacobson radical of the endomorphism ring of M is nilpotent. Our purpose here is to present a generalization of this result.

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The methods used are a straightforward adaptation of those used by Boyle and Feller [4] in their work on modules with Krull dimension. Indeed, one of our main objectives here is to show how much of the work of Boyle, Feller, and their students can be profitably and readily extended to more general situations.

1. Full modules

Let $\tau \in R$-tors. By [7, proposition 1.7] we know that if N is a τ-dense submodule of a τ-torsionfree left R-module M then N must be large in M. If the converse is also true, then we say that the left R-module M is *τ-full*. That is to say, a τ-torsionfree left R-module M is τ-full if and only if its τ-dense submodules are precisely those submodules large in it. This notion was first studied by Boyle and Feller [1, 2, 6] under the name of "the large condition" in connection with their research on modules with Krull dimension (their definition is not phrased as above; for the link between Krull dimension and torsion theories in its most general form, see chapter 14 of [8]); it was generalized to arbitrary torsion theories by Lau [12], who noted some of the basic properties of full modules. In particular, Lau proved the following results:

(1.1) PROPOSITION. *If $\tau \in R$-tors and if N is a submodule of a left R-module M then:*

(1) *If M is τ-full, so is N;*

(2) *If M is τ-full and N is τ-pure in M, then M/N is τ-full;*

(3) *If M is τ-torsionfree and if N is τ-full and τ-dense in M, then M is τ-full.*

(1.2) PROPOSITION. *If $\tau \in R$-tors and if M is a τ-torsionfree left R-module then the sum of an arbitrary family of τ-full submodules of M is again τ-full.*

It is clear that τ-cocritical left R-modules are τ-full and, indeed, a nonzero uniform τ-torsionfree left R-module is τ-full if and only if it is τ-cocritical. More generally, let us call a nonzero left R-module M *τ-semicocritical* if and only if it can be embedded in a direct sum of finitely-many τ-cocritical left R-modules. (Again, this is a generalization due to Lau of a notion first proposed by Boyle and Feller [1, 2] in connection with their work on modules having Krull dimension.) Clearly τ-semicocritical left R-modules are τ-full and have finite uniform dimension. Indeed, the converse is also true: a nonzero left R-module is τ-semicocritical if and only if it is τ-full and has finite uniform dimension.

(1.3) PROPOSITION. *Let $\tau \in R$-tors. If M is a τ-torsionfree τ-noetherian left R-module then M is τ-full if and only if it is τ-semicocritical.*

PROOF. It suffices to show that M has finite uniform dimension. Indeed, assume that $\{N_i \mid i \geqq 1\}$ is an infinite set of nonzero submodules of M the sum of which is direct. For each positive integer k, let W_k be the τ-purification of $\bigoplus_{i=1}^{k} N_i$ in M. Then W_k is a τ-pure submodule of M and we have $W_1 \subseteq W_2 \subseteq \cdots$. Since M is τ-noetherian, there exists an integer t satisfying $W_t = W_{t+1} = \cdots$. If $x \in N_{t+1}$ then, in particular, $x \in W_{t+1} = W_t$ and so there exists a τ-dense left ideal I of R satisfying $Ix \subseteq \bigoplus_{i=1}^{t} N_i$. But we also have $Ix \subseteq N_{t+1}$ and so, by the directness of the sum $\sum_{i=1}^{t+1} N_i$, we must have $Ix = (0)$. Since M is τ-torsionfree, this implies that $x = 0$ and so $N_{t+1} = (0)$, which is a contradiction. Thus M must have finite uniform dimension. □

In particular, if M is a τ-torsionfree left R-module which has a τ-composition series in the sense of [11] then M is τ-full if and only if it is τ-semicocritical. If M is a τ-semicocritical left R-module then there exists a large τ-dense submodule of M of the form $\bigoplus_{i=1}^{k} N_i$, where the N_i are τ-cocritical submodules of M [12, 14]. Thus, if $\mathscr{C}(\tau)$ is the class of all τ-cocritical left R-modules then the τ-semicocritical left R-modules are those which are $\mathscr{C}(\tau)$-finite-dimensional in the sense of [13].

If $\tau \in R$-tors and if M is any left R-module then we can define the submodule $F_\tau(M)$ of M by

$$F_\tau(M)/T_\tau(M) = \sum \{\bar{N} \subseteq M/T_\tau(M) \mid \bar{N} \text{ is } \tau\text{-full}\}.$$

If $\alpha : M \to M'$ is an R-homomorphism of left R-modules then $T_\tau(M)\alpha \subseteq T_\tau(M')$ and so α induces an R-homomorphism $\bar{\alpha} : M/T_\tau(M) \to M'/T_\tau(M')$. By Proposition 1.1(2), the image of $F_\tau(M)/T_\tau(M)$ under $\bar{\alpha}$ is contained in $F_\tau(M')/T_\tau(M')$ and so $F_\tau(M)\alpha \subseteq F_\tau(M')$. Thus $F_\tau(-)$ is a subfunctor of the identity functor on R-mod, which is clearly idempotent. Moreover, by Proposition 1.2 we see that if N is a submodule of a left R-module M then $F_\tau(N) = F_\tau(M) \cap N$, so the function $F_\tau(-)$ is left exact. Therefore we can associate with $F_\tau(-)$ a torsion subfunctor $\bar{F}_\tau(-)$ of the identity functor on R-mod obtained by a Loewy-sequence construction as described in section 2 of [7]. In particular, we define subfunctors $F_\tau^{(i)}(-)$ of the identity functor on R-mod such that, for any left R-module M,

(1) $F_\tau^{(0)}(M) = (0)$;

(2) If i is not a limit ordinal then $F_\tau^{(i)}(M)/F_\tau^{(i-1)}(M) = F_\tau(M/F_\tau^{(i-1)}(M))$;

(3) If i is a limit ordinal then $F_\tau^{(i)}(M) = \bigcup\{F_\tau^{(j)}(M) \mid j < i\}$.

For each left R-module M there exists a least ordinal $h = h(\tau, M)$ satisfying $F_\tau^{(h)}(M) = F_\tau^{(h+1)}(M) = \cdots$ and we set $\bar{F}_\tau(M) = F_\tau^{(h)}(M)$. The ordinal h is called the *F_τ-length* of M.

(1.4) Proposition. *Let $\tau \in R$-tors. Any τ-noetherian left R-module has finite F_τ-length.*

Proof. Let M be a τ-noetherian left R-module having F_τ-length h. For each $i \leqq h$, let N_i be the τ-purification of $F_\tau^{(i)}(M)$ in M. Then $F_\tau^{(i)}(M) \subseteq N_i \subseteq F_\tau^{(i+1)}(M)$ for each such i. In particular, we have an ascending chain $N_0 \subseteq N_1 \subseteq \cdots$ of τ-pure submodules of M and so there must exist a natural number k such that $N_k = N_{k+1} = \cdots$ This implies that $F_\tau^{(k)}(M) = F_\tau^{(k+1)}(M) = \cdots$ and so $h \leqq k$. □

By [7, proposition 2.6] there exists a torsion theory $\tau' \in R$-tors satisfying the condition that $\bar{F}_\tau(-) = T_{\tau'}(-)$. What is this torsion theory like? In chapter 12 of [8] we introduced the notion of the Gabriel filtration associated with a quasi-dimension function defined on a complete lattice and with an element of that lattice. For the special case of the lattice R-tors and an element τ of it, the construction described there yields a transfinite ascending sequence of torsion theories

$$\tau = \tau_0 \leqq \tau_1 \leqq \tau_2 \leqq \cdots,$$

called the *Gabriel filtration* of τ. In particular, for each ordinal i we have

$$\tau_{i+1} = \tau_i \vee \{\xi(M) \mid M \text{ is } \tau_i\text{-cocritical}\}.$$

It is thus clear that $\tau_{i+1} \leqq (\tau_i)'$ for each ordinal i. If M is a τ-noetherian left R-module then M is τ_i-noetherian for each ordinal i.

(1.5) Proposition. *Let $\tau \in R$-tors and let M be a τ-noetherian left R-module. Then M is τ_{i+1}-torsion if and only if it is $(\tau_i)'$-torsion.*

Proof. If M is τ_{i+1}-torsion then, by what we have already observed, it is $(\tau_i)'$-torsion. Now assume that it is $(\tau_i)'$-torsion and let $N = T_{\tau_i}(M)$. Then M/N is τ_i-noetherian and τ_i-torsionfree. Moreover, by definition of $(\tau_i)'$, it is τ_i-full. By Proposition 1.3 we then know that M/N is τ_i-semicocritical and hence, clearly τ_{i+1}-torsion. Since N is also τ_{i+1}-torsion, this implies that M is τ_{i+1}-torsion. □

One of the major uses of the Gabriel filtration of a torsion theory τ is to define the notion of the *τ-dimension* of a module: a left R-module M has τ-dimension if and only if it is τ_i-torsion for some ordinal i. The special case of $\tau = \xi$ (where ξ is the unique minimal element of R-tors) is also given in [7]. Indeed, the ξ-dimension of a left R-module is just its Gabriel dimension. The general case is considered in [8]. In [10], Gabriel filtrations are used to discuss the local cohomology of noncommutative rings; in [9] they are used in connection with a partial solution of Boyle's Conjecture.

If $\tau \in R$-tors and if M is a left R-module having τ-dimension then we have an ascending sequence

$$T_{\tau_0}(M) \subseteq T_{\tau_1}(M) \subseteq \cdots \subseteq T_{\tau_h}(M) = M$$

of τ-pure submodules of M. If the module M is τ-noetherian then only finitely-many of these inclusions can be proper. That is to say, under such circumstances there exists a finite sequence of ordinals $\langle n(0), \cdots, n(k)\rangle$ such that

(1) $n(0) = 0$;

(2) If $0 \leqq i < k$ then $n(i+1) = \inf\{j > n(i) \mid T_{\tau_j}(M) \neq T_{\tau_{n(i)}}(M)\}$;

(3) $T_{\tau_{n(k)}}(M) = M$.

We will say that the module M is of *τ-type* $\langle n(0), \cdots, n(k)\rangle$. Note that none of the ordinals $n(i)$ can be a limit ordinal.

2. Endomorphisms with large kernels

Let M be a left R-module with endomorphism ring S. Then $L(S) = \{\alpha \in S \mid \ker(\alpha) \text{ is large in } M\}$ is a two-sided ideal of S. Indeed, if M is quasi-injective then $L(S)$ is just the Jacobson radical of S [5]. In this section we will be interested in the behavior of $L(S)$ when M is a τ-noetherian left R-module having τ-dimension for some torsion theory τ in R-tors.

(2.1) Proposition. *Let $\tau \in R$-tors and let M and N be τ-torsionfree left R-modules. If $\alpha : M \to N$ is an R-homomorphism the kernel of which is large in M then $F_\tau(M) \subseteq \ker(\alpha)$.*

Proof. Since M is τ-torsionfree, we know that $F_\tau(M)$ is τ-full by Proposition 1.2. Moreover, $\ker(\alpha) \cap F_\tau(M)$ is large in $F_\tau(M)$ and so is τ-dense there. Since $F_\tau(M)\alpha$ is τ-torsionfree, this implies that we must in fact have $F_\tau(M)\alpha = 0$. □

(2.2) Proposition. *Let $\tau \in R$-tors and let M be a τ-torsionfree left R-module having endomorphism ring S. Then $F_\tau^{(i)}(M)L(S)^i = (0)$ for each positive integer i.*

Proof. By Proposition 2.1 we see that $F_\tau(M)L(S) = 0$. Let N_1 be the τ-purification of $F_\tau(M)$ in M. Then $F_\tau(M) \subseteq N_1 \subseteq F_\tau^{(2)}(M)$. If $\alpha \in L(S)$ then $F_\tau(M)\alpha = (0)$ so α induces an R-homomorphism $\bar{\alpha} : M/F_\tau(M) \to M$. Since M is τ-torsionfree, $N_1/F_\tau(M) \subseteq \ker(\bar{\alpha})$ and so $\bar{\alpha}$ induces an R-homomorphism $\bar{\bar{\alpha}} : M/N_1 \to M$. Since M/N_1 is τ-torsionfree, $F_\tau^{(2)}(M)/N_1$ is τ-full by Proposition 1.2 and so, by Proposition 1.1(2), $F_\tau^{(2)}(M)\alpha = [F_\tau^{(2)}(M)/N_1]\bar{\bar{\alpha}}$ is also τ-full. Hence $F_\tau^{(2)}(M)\alpha \subseteq F_\tau(M)$. Thus $F_\tau^{(2)}(M)\alpha L(S) = (0)$. This implies that $F_\tau^{(2)}(M)L(S)^2 = (0)$. Continue in this manner to obtain the desired result. □

(2.3) PROPOSITION. *Let $\tau \in R$-tors and let M be a τ-torsionfree τ-noetherian left R-module with endomorphism ring S having τ-dimension and of τ-type $\langle n(0), \cdots, n(k) \rangle$. For each $0 \leqq i < k$, let $\sigma_i = \tau_{n(i+1)-1}$ and let $h(i)$ be the F_{σ_i}-length of M. Then $L(S)$ is a nilpotent ideal of S the index of nilpotency of which is no greater than the sum of the nonleading coefficients of the polynomial $\Pi_{i=0}^{k-1} [X - h(i)]$.*

PROOF. For each $0 \leqq i < k$, set $M_i = T_{\tau_{n(i)}}(M)$. Then we have a chain

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_k = M$$

of submodules of M, where

$$(1) \qquad M_i = T_{\sigma_i}(M), \qquad \text{and}$$

$$(2) \qquad M_{i+1} \text{ is the } \tau_{n(i+1)}\text{-purification of } M_i \text{ in } M$$

for all $0 \leqq i < k$.

Since M is τ-noetherian, it is τ_j-noetherian for all ordinals j. Therefore, by Proposition 1.5,

$$(3) \qquad M_{i+1} = \bar{F}_{\sigma_i}(M)$$

for all $0 \leqq i < k$. By Proposition 1.4, we see that for each such i the F_{σ_i}-length of M is a positive integer $h(i)$. Therefore

$$(4) \qquad M_{i+1} = F_{\sigma_i}^{(h(i))}(M)$$

for all $0 \leqq i < k$.

For each $1 \leqq j \leqq h(i)$, let $M_{i,j} = F_{\sigma_i}^{(j)}(M)$ and set $M_{i,0} = M_i$. This yields a chain

$$M_i = M_{i,0} \subseteq M_{i,1} \subset \cdots \subset M_{i,h(i)} = M_{i+1}$$

of submodules of M. We want to establish the following claim:

CLAIM. If $0 \leqq i < k$ and if $M_i L(S)^p = (0)$ for some natural number p then $M_{i,j} L(S)^{jp+j+p} = (0)$ for all $0 \leqq j \leqq h(i)$.

Clearly this is true for $j = 0$. Now assume inductively that $j \geqq 0$ and that the claim has already been established for j. Let α be an element of $L(S)^{jp+j+p}$. Then $M_{i,j}\alpha = (0)$. If $x \in M_{i,j+1}$ then $Rx\alpha$ is a homomorphic image of the σ_i-full left R-module $\bar{R}\bar{x} = [Rx + W]/W$, where W is the σ_i-purification of $M_{i,j}$ in M. Thus, if $\bar{K} = [\ker(\alpha) \cap (Rx + W)]/W$ is large in $\bar{R}\bar{x}$ then $Rx\alpha$ is σ_i-torsion and so is contained in $M_i = T_{\sigma_i}(M)$. This implies, in particular, that $Rx\alpha L(S) \subseteq M_i$. Now assume that $\bar{K}$ is not large in $\bar{R}\bar{x}$ and let $\bar{D}$ be its relative complement there. The

R-homomorphism α canonically induces an R-homomorphism $\bar{\alpha}: \bar{R}\bar{x} \to M$, the kernel of which is $\bar{K}$. Thus $(\bar{D}+\bar{K})\bar{\alpha} \cong \bar{D}$. But $\bar{D}$ is σ_i-full and so $(\bar{D}+\bar{K})\alpha \subseteq M_{i,1}$. Let $\beta \in L(S)$. Then $M_{i,1}\beta \subseteq M_i$, since $\ker(\beta) \cap M_{i,1}$ is large in $M_{i,1}$ and hence is σ_i-dense there. Thus $(\bar{D}+\bar{K})\bar{\alpha}\beta \subseteq M_i$. Since $\bar{R}\bar{x}$ is σ_i-full and $\bar{D}+\bar{K}$ is a large submodule of it, we see in particular that $\bar{D}+\bar{K}$ is σ_i-dense in $\bar{R}\bar{x}$ and so $(\bar{D}+\bar{K})\bar{\alpha}\beta$ is σ_i-dense in $Rx\alpha\beta$. Thus M_i is σ_i-dense in $Rx\alpha\beta + M_i$. But M_i is σ_i-pure in M and hence in $Rx\alpha\beta + M_i$, so we must have $Rx\alpha\beta + M_i = M_i$. In other words, $Rx\alpha\beta \subseteq M_i$. This is true for any such β and so, again, we have $Rx\alpha L(S) \subseteq M_i$. Since this is true for any $x \in M_{i,j+1}$ and all $\alpha \in L(S)^{jp+j+p}$, we have $M_{i,j+1}L(S)^{jp+j+p+1} \subseteq M_i$. By hypothesis, $M_iL(S)^p = (0)$ and so

$$M_{i,j+1}L(S)^{(j+1)p+(j+1)+p} = M_{i,j+1}L(S)^{jp+j+p+1}L(S)^p \subseteq M_iL(S)^p = (0).$$

This establishes the claim.

By Proposition 2.2, we have, in particular, that $M_1L(S)^{h(0)} = (0)$. Since $M_2 = M_{1,h(1)}$, the above claim implies that $M_2L(S)^{h(0)h(1)+h(0)+h(1)} = (0)$, where $h(0)h(1)+h(0)+h(1)$ is precisely the sum of the nonleading coefficients of the polynomial $[X+h(0)][X+h(1)]$. Continue in this manner to prove the proposition. □

(2.4) Corollary. *Let $\tau \in R$-tors and let M be a τ-torsionfree τ-noetherian left R-module having finite uniform dimension. Then nil subrings of the endomorphism ring of M are nilpotent.*

Proof. This follows from Proposition 2.3 and from [17, theorem 3]. □

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